Indian Journal of Applied Business and Economic Research
Vol. 3, No. 2, 2022 : pp. 189-205

URL : www.arfjournals.com
https://DOI:10.47509/IJABER.2022.v03i02.01

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

GENERATING FUNCTION OF RUNS OF A FIXED LENGTH OR MORE UNTIL A STOPPING TIME FOR HIGHER ORDER MARKOV CHAIN

Anuradha

Lady Shri Ram College for Women, Lajpat Nagar, New Delhi India. E-mail: sarkar.anuradha@gmail.com

Received : 03 July 2022; Revised : 13 August 2022; Accepted : 29 August 2022; Published : 27 December 2022

Abstract: Let $\{X_j : j \geq -m + 1\}$ be a $\{0, 1\}$ valued homogeneous Markov chain of order m. For $i = 0, -1, \ldots, -l + 1$, we will set $R_i = 0$. Now if we define $R_i = \prod_{j=i-1}^{i-1}(1-R_j)\prod_{j=i}^{i+k-1} X_j$ then $R_i = 1$ implies that an l-look-back run of length k has occurred starting at i. Here R_i is defined inductively as a run of 1's starting at i, provided that no l-look-back run of length k occurs, starting at time $i-1, i-2, \ldots, i-l$. We obtain the conditional distribution of the number of runs of a fixed length at least k_1 until the stopping time i.e. the r-th occurrence of the l-look-back run of length k where $k_1 \leq k$. and it's probability generating function. The number of runs of length at least k_1 until the stopping time has been expressed as the sum of r independent random variables with the first random variable having a slightly different distribution.

Keywords: Markov chain, runs, stopping time, probability generating function, strong Markov property.

To cite this article:

Anuradha (2022). Generating Function of Runs of a Fixed Length or More Until a Stopping Time for Higher order Markov Chain. *Indian Journal of Applied Business and Economic Research*, Vol. 3, No. 2, pp. 189-205. https://DOI: 10.47509/IJABER.2022.v03i02.01

1 Introduction

Feller [1968] introduced the concept of runs as an example of a renewal event and since then the theory of distributions of runs has been developed extensively by researchers. In order to study various run related statistics in details, many techniques such as method of conditional p.g.f.s (see Ebneshahrashoob & Sobel [1990]) and Markov embedding technique (see Fu and Koutras [1994]) etc have been developed and applied effectively.

We consider the set up of an m-th order homogeneous $\{\mathbf{0},\mathbf{1}\}$-valued Markov chain and we assume that the initial condition $\{X_0 = x_0, X_{-1} = x_1, \dots, X_{-m+1} = x_{m-1}\}$ is given. The random variable X_n assumes value $\mathbf{1}$ when the outcome of the n-th trial is success and value $\mathbf{0}$ when it is failure. A run of length k is defined as a consecutive occurrence of k successes.

The marginal distributions of number of failures, successes and success-runs of length less than k until the first occurrence of consecutive k successes when the underlying random variables are either i.i.d. or homogeneous Markov chain or binary sequence of order k was studied by Aki and Hirano [1994]. Under the similar set up, Aki and Hirano [1995] derived the joint distributions of number of failures, successes and runs of success. Hirano et. al. [1997] obtained interesting results related to the distributions of number of success-runs of length l until the first occurrence of the success-run of length k for an m-th order homogeneous Markov chain where $m \leq l < k$ under various counting schemes like runs of length k_1, overlapping runs of length k_1, non-overlapping runs of length k_1 etc. The joint distributions of the waiting time and the number of outcomes such as failures, successes and success-runs of length less than k for various counting schemes of runs under the set up of an m-th order homogeneous Markov chain was explored by Uchida [1998]. Martin [2005] obtained the distribution of the number of successes in success runs of length at least k for a higher order Markovian sequence.

The l-look-back counting scheme for runs was introduced by Anuradha [2022a]. In this scheme, if a run has been counted starting at time i, i.e., $\{X_i = X_{i+1} = \cdots = X_{i+k-1} = \mathbf{1}\}$, then no runs can be counted till the time point $i+l$ and the next counting of runs can start only from the time point $i + l + 1$. This scheme is repeated every time a run is counted. In other words, if a run is counted starting at time i, there are k-consecutive successes from the time point i and no runs of length k can be counted which start at the time points $i-1, i-2, \dots, i-l$. Clearly, if $l = 0$, this counting scheme of run matches exactly with the counting of overlapping runs of length k, while if we consider $l = k-1$, this counting scheme results in the counting of non-overlapping runs of length k. Under the set up of m-th order homogeneous Markov chain, Anuradha [2022a] established that the waiting time distribution of the n-th occurrence of the l-look-back run of length k converges to an extended Poisson distribution when the system exhibits strong propensity towards success. Under the same set up, central limit theorem was established for the number of l-look-back runs of length k till the n-th trial. Anuradha [2022b] obtained the conditional distribution of the number of runs of length exactly k_1 till the r-th occurrence of l-look-back run of length k when the underlying random variables follow an m-th order Markov chain and identified the form of conditional distribution.

In study of DNA sequences, tandem repeats are short lengths of DNA that are repeated multiple times within a gene and repetitions are directly adjacent to each other. Due to the significance of tandem repeats in the biological studies, much work has been devoted to developing algorithms for their detection. A statistic based on the number of runs of length at least k_1 along with the related distributions would be useful in studying tandem repeats . But due to complex dependency structure, it is often very difficult to find the exact distributions of number of runs of length at least k_1. Therefore, one can obtain approximate distributions of such statistic to make meaningful statistical conclusions.

In this paper, we take a different point of view and study the distribution of number of runs of length at least k_1 until a stopping time. Surprisingly, this turns out to be much simpler and can be identified through well known distributions, namely Geometric and Binomial distribution. The stopping time that we consider is the r-th occurrence of the l-look-back run of length k, where $k_1 \leq k$. As r increases, the stopping times will approach ∞. Hence, this may be used to get an alternate approximation for the distribution of number of runs of length at least k_1. For example, one may try and find the law of large numbers or a central limit theorem for the number of runs of length at least k_1 using the conditional distribution.

The next section specifies the important definitions and the main theorem and corollaries related to the distribution of the number of runs of length at least k_1 until the r-th occurrence of the l-look-back run of length k where $k_1 \leq k$. Section 3 formalises the basic set up for deriving the results. Section 4 is devoted to the proof of the main Theorem. The conditional probability generating function method has been employed to prove the main result.The problem has been recast into a first order homogeneous Markov chain taking values in a finite set and strong Markov property has been used to derive the recurrence relation involving the probabilities of the number of runs of length at least k_1 until the stopping time. This relation is used to establish the recurrence relation of the probability generating functions which is solved to obtain the explicit expression.

2 Definitions and Statement of Results

Let $X_{-m+1}, \ldots, X_0, X_1, \ldots$ be a sequence of stationary m-order $\{\mathbf{0}, \mathbf{1}\}$ valued Markov chain. Assume that the states of $X_{-m+1}, \ldots, X_0$ are known i.e., $x_0, x_{-1}, \ldots, x_{-m+1}$ are known and we take the initial state as $X_0 = x_0, X_{-1} = x_{-1}, \ldots, X_{-m+1} = x_{-m+1}\}$.

Define the set $A_i = \{0, 1, \ldots, 2^i - 1\}$ for any $i \geq 0$. It is clear that A_i and $\{\mathbf{0}, \mathbf{1}\}^i$ can be connected by the mapping $x = (x_0, x_1, \ldots, x_{i-1}) \longrightarrow \sum_{j=0}^{i-1} 2^j x_j$. Since, $\{X_n : n \geq -m+1\}$ is m^{th} order Markov chain, we have the transition probabilities

$$p_x = \mathbb{P}(X_{n+1} = 1 | X_n = x_0, X_{n-1} = x_1, \ldots, X_{n-m+1} = x_{m-1}) \tag{1}$$

where $x = \sum_{j=0}^{m-1} 2^j x_j \in A_m$, for any $n \geq 0$. Therefore, we have $q_x = \mathbb{P}(X_{n+1} = 0 | X_n = x_0, X_{n-1} = x_1, \ldots, X_{n-m+1} = x_{m-1}) = 1 - p_x$. We assume that $0 < p_x < 1$ for all $x \in A_i$.

Definition 1 *(***l-look-back run***) (Anuradha [2022a]) Fix two integers $k \geq 1$ and $1 \leq l \leq k-1$. We set $R_i(k,l) = 0$ for $i = 0, -1, \ldots, -l+1$ and for any $i \geq 1$, define inductively,*

$$R_i(k,l) = \prod_{j=i-1}^{i-l} (1 - R_j(k,l)) \prod_{j=i}^{i+k-1} X_j. \tag{2}$$

If $R_i(k,l) = 1$, we say that a l-look-back run of length k *has been recorded which started at time i.*

It should be noted that for an l-look-back run to start at the time point i, we need to look back at the preceding l many time points, i.e., $i-1$ to $i-l$, none of which can be the starting point of an l-look-back run of length k.

Next we define the stopping times where the r-th l-look-back run of length k is completed.

Definition 2 *(Anuradha [2022b]) For $r \geq 1$, the stopping time $\tau_r(k,l)$ be the (random) time point at which the r-th l-look-back run of length k is completed. In other words,*

$$\tau_r(k,l) = \inf\{n : \sum_{i=1}^{n} R_i(k,l) = r + k - 1\}. \tag{3}$$

Before we introduce runs of length at least k, we need the concept of run of length exactly k.

Definition 3 *(Runs of length exactly k) (see Anuradha [2022b]) When $k(\geq 1)$ consecutive successes, either occur at the beginning of the sequence or end of the sequence or bordered on both sides by failures, contribute towards the counting of a run then we call it a run of length exactly k.Note that when there are more than k consecutive successes then it is not counted as run of length exactly k.*

We may represent this mathematically as follows:

$$R_i^{(E;k)} = \begin{cases} \prod_{j=1}^{k} X_j (1 - X_{k+1}) & \text{if } i = 1 \\ (1 - X_{i-1}) \prod_{j=i}^{i+k-1} X_j (1 - X_{i+k}) & \text{if } 1 < i < n - i + 1 \\ (1 - X_{n-k}) \prod_{j=n-k+1}^{n} X_j & \text{if } i = n - k + 1. \end{cases}$$

Note here that $R_i^{(E;k)} = 1$ if and only if a run of length exactly k starts at time point i. Anuradha [2022b] studied the distribution of runs of length exactly k_1 until the stopping time $\tau_r(k,l)$.

Now, we define the runs of length at least k as follows:

Definition 4 *A run of length at least k starts at time point $i \geq 1$, if at least one run of length exactly q starts at i for some $q \geq k$.*

This may be mathematically stated as follows: for $i \geq 1$, set

$$S_i^{(L;k)} = \sum_{q=k}^{n} R_i^{(E;q)}. \tag{4}$$

It should be noted that $S_i^{(L;k)} = 1$ if and only if a run of length at least k starts at i.

In this paper, we study the number of runs of length at least k_1 till the stopping time $\tau_r(k,l)$ (see Definition (2). Fix any constant $k_1 \leq k$. For each $r \geq 1$, we define the random variable

$$N_r^{(L)}(k_1) := N_{\tau_r(k,l)}^{(L)}(k_1) = \sum_{i=1}^{\tau_r(k,l)} S_i^{(L;k)} \tag{5}$$

as the number of runs of length at least k_1 until the stopping time $\tau_r(k,l)$.

Let us consider the following example to understand the concepts. The following sequence of **0**'s and **1**'s of length 30 was observed:

110101110111110111010110110111.

For $k = 3$ and $l = 1$, it should be noted that, $R_6(3,1) = R_{10}(3,1) = R_{12}(3,1) = R_{16}(3,1) = R_{28}(3,1) = 1$, while for other values of i, $R_i(3,1) = 0$. Thus, stopping times are given by $\tau_1(3,1) = 8, \tau_2(3,1) = 12, \tau_3(3,1) = 14, \tau_4(3,1) = 18$ and $\tau_5(5,1) = 30$. For $k_1 = 2$, we observe that values of $S_1^{(L;2)} = S_6^{(L;2)} = S_{10}^{(L;2)} = S_{16}^{(L;2)} = S_{22}^{(L,2)} = S_{25}^{(L,2)} = S_{28}^{(L,2)} = 1$ and $S_j^{(L;2)} = 0$ for other values of j. Now the number of runs of length at least 2 until the r-th occurrence of 1-look-back run of length 3 can be easily found from the above. Indeed, for $r = 1$, we have $\tau_1(3,1) = 8$ and hence $N_1^{(L)}(k_1) = \sum_{i=1}^{\tau_1(3,1)} S_i^{(L,2)} = 2$. Similarly, $N_2^{(L)}(k_1) = 3 = N_3^{(L)}(k_1), N_4^{(L)}(k_1) = 4$ and $N_5^{(L)}(k_1) = 7$.

We will use another random time T which we have introduced in (7). Though the formal definition is provided later, this is time of the first occurrence of k_1-many consecutive successes. In the above example (for $k_1 = 2$), it is clear that first occurrence of 2 consecutive successes is observed at time $T = 2$.

Now, we define the probability generating function of $N_r^{(L)}(k_1)$. Let us set

$$\zeta_r^{(L)}(s;k_1) := \sum_{n=0}^{\infty} \mathbb{P}(N_r^{(L)}(k_1) = n)s^n. \tag{6}$$

Now, we will state our main result.

Theorem 1 *For any initial condition $x \in A_i$ and $k_2 = k - k_1$ and $k_1 \geq m$, the probability generating function of $N_r^{(L)}(k_1)$ is given by*

$$\zeta_r^{(L)}(s;k_1) = \left[\frac{\left(p_{2^m-1}\right)^{k_2} s}{1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s}\right] \left[\left(p_{2^m-1}\right)^{l+1} + \frac{\left(p_{2^m-1}\right)^{k_2} s}{1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)\right]^{r-1}.$$

The result provides a powerful representation of $N_r^{(L)}(k_1)$.

Corollary 1 *Let* $\{G_L^{(i)} : i = 1, \ldots, r\}$ *and* $\{B_L^{(i)} : i = 1, \ldots, r\}$ *be two independent sets of random variables with each* $G_L^{(i)}$ *having a geometric distribution (taking values in* $\{0, 1, \ldots, \}$*) with parameter* $\big(p_{2^m-1}\big)^{k_2}$ *and each* $B_L^{(i)}$ *having a Bernoulli distribution with parameter* $\Big(1 - \big(p_{2^m-1}\big)^{l+1}\Big)$*, then we have*

$$N_r^{(L)}(k_1) \overset{d}{=} \big(1 + G_L^{(1)}\big) + \sum_{i=2}^{r} \big(1 + G_L^{(i)}\big) B_L^{(i)}.$$

Indeed, it is easy to see that generating function of $G_L^{(i)}$, for $i \geq 1$, is given by

$$\frac{\big(p_{2^m-1}\big)^{k_2}}{1 - \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big)s}$$

and the generating function of $B_L^{(i)}$, for $i \geq 1$, is given by

$$\big(p_{2^m-1}\big)^{l+1} + s\Big(1 - \big(p_{2^m-1}\big)^{l+1}\Big).$$

Therefore, the generating function of $\big(1 + G_L^{(i)}\big) B_L^{(i)}$ is given by

$$\big(p_{2^m-1}\big)^{l+1} + \frac{\big(p_{2^m-1}\big)^{k_2} s}{1 - \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big)s}\Big(1 - \big(p_{2^m-1}\big)^{l+1}\Big).$$

From the independence of $G_L^{(i)}$ and $B_L^{(i)}$ for $i \geq 1$, the corollary easily follows.

If we set $k_1 = 1$, i.e., $k_2 = k - 1$, then $N_r^{(L)}(k_1)$ represents the total number of success runs until $\tau_r(k, l)$.

Corollary 2 *For the i.i.d. case or the Markov dependent case, the probability generating function of the number of success runs till the* r*-th occurrence of the* l*-look-back run of length* k*, i.e.,* $N_r^{(L)}(1)$ *is given by*

$$\zeta_r^{(L)}(s; 1) = \left[\frac{\big(p_{2^m-1}\big)^{k-1} s}{1 - \Big(1 - \big(p_{2^m-1}\big)^{k-1}\Big)s}\right]\left[\big(p_{2^m-1}\big)^{l+1} + \frac{\big(p_{2^m-1}\big)^{k-1} s}{1 - \Big(1 - \big(p_{2^m-1}\big)^{k-1}\Big)s}\Big(1 - \big(p_{2^m-1}\big)^{l+1}\Big)\right]^{r-1}.$$

3 Set-up

Now we outline the underlying set up which will be used in the subsequent sections to establish the results. Let us define two functions $f_0, f_1 : A_{k_1} \to A_{k_1}$ by

$$f_1(x) = 2x + 1 \pmod{2^{k_1}} \text{ and } f_0(x) = 2x \pmod{2^{k_1}}.$$

Further define a projection $\theta_m : A_{k_1} \to A_m$ by $\theta_m(x) = x \pmod{2^m}$. Now, set $X_{-m} = X_{-m-1} = \cdots = X_{-k_1+1} = 0$. Define a sequence of random variables $\{Z_n : n \geq 0\}$ as follows:

$$Z_n = \sum_{j=0}^{k_1-1} 2^j X_{n-j}.$$

Since $X_i \in \{\mathbf{0}, \mathbf{1}\}$ for all i, Z_n assumes values in the set A_{k_1}. The random variables X_n's are stationary and forms a m^{th} order Markov chain, hence we have that $\{Z_n : n \geq 0\}$ is a homogeneous Markov chain with transition matrix given by

$$\mathbb{P}(Z_{n+1} = y | Z_n = x) = \begin{cases} p_{\theta_m(x)} & \text{if } y = f_1(x) \\ 1 - p_{\theta_m(x)} & \text{if } y = f_0(x) \\ 0 & \text{otherwise.} \end{cases}$$

Note that Z_n is even if and only if $X_n = 0$. This motivates us to define the function $\kappa : A_{k_1} \to \{0, 1\}$ by

$$\kappa(x) = \begin{cases} 1 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even.} \end{cases}$$

Therefore, $\kappa(Z_n) = 1$ if and only if $X_n = 1$. Hence, the definition of l-look-back run can be described in terms of Z_n's as

$$R_i(k, l) = \prod_{j=i-l}^{i-1} (1 - R_j(k, l)) \prod_{j=i}^{i+k-1} \kappa(Z_j).$$

Let us fix any initial condition $x \in A_m$. We denote the probability measure governing the distribution of $\{Z_n : n \geq 1\}$ with $Z_0 = x \in A_k$ by $\mathbb{P}_x$. Since we have set $X_{-m} = X_{-m-1} = \cdots = X_{-k+1} = 0$, we have $Z_0 = x$.

In order to obtain the recurrence relation for the probabilities, we will condition the process after the first occurrence of the run of length k_1. Therefore, we consider the stopping time T when the first occurrence of a run of length k_1 ends, i.e., when we observe k successes consecutively for the first time. More precisely, define

$$T := \inf\{i \geq k_1 : \prod_{j=i-k_1+1}^{i} X_j = 1\}. \tag{7}$$

We would like to translate the above definition to Z_i's. It must be the case that when T occurs, last k_1 trials have resulted in success, which may be described by $\kappa(Z_j) = 1$ for

$j = i - k_1 + 1$ to i. Therefore, Z_T must equal $2^{k_1} - 1$. Since this is the first occurrence, this has not happened earlier. So, T can be better described as

$$T = \inf\{i \geq k_1 : Z_i = 2^{k_1} - 1\},$$

i.e., the first visit of the chain to the state $2^{k_1} - 1$ after time $k_1 - 1$. Now, we note that $\{Z_n : n \geq 0\}$ is a Markov chain with finite state space. Further, since $0 < p_u < 1$ for $u \in A_m$, this is an irreducible chain; hence, it is positive recurrent. So we must have $\mathbb{P}_x(T < \infty) = 1$. We observe that when the first occurrence of k consecutive successes happen, we must have the occurrence of k_1 successes previously since $k_1 \leq k$. Therefore, we have $\mathbb{P}_x(T < \tau_1(k,l)) = 1$.

4 Number of runs of length at least k_1

In this section, we study the runs of length at least k_1. Define the probability, for $x \in A_m, n \in \mathbb{Z}$,

$$g_r^{(L,x)}(n) = \mathbb{P}_x\big(N_r^{(L)}(k_1) = n\big). \tag{8}$$

We note that since $N_r^{(L)}(k_1) \geq 1$, $\mathbb{P}_x\big(N_r^{(L)}(k_1) = n\big) = 0$ for $n \leq 0$.

Next we obtain a recurrence relation between $g_1^{(L,x)}(n; k_1)$.

Theorem 2 *For $n \geq 0$ and $x \in A_m$, we have*

$$g_1^{(L,x)}(n; k_1) = \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big) g_1^{(L,2^m-2)}(n-1; k_1) + \big(p_{2^m-1}\big)^{k_2} \mathbb{I}_n(1) \tag{9}$$

where $\mathbb{I}_{v_1}(v_2)$ is the indicator function defined by

$$\mathbb{I}_{v_1}(v_2) = \begin{cases} 1 & \text{if } v_1 = v_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof : If $k_2 = 0$, i.e., $k = k_1$, we must have have $\mathbb{P}_x(N_1^{(L)}(k) = 1) = 1$. Clearly, $g_1^{(L,x)} = 1$ for all $x \in A_m$ satisfy the recurrence relation in (9).

Now, for the case when $k_2 = k - k_1 > 0$, first note that the occurrence of a look-back run of length k, implies that a run of length at least k_1 must have occurred. Therefore, $N_1^{(L)}(k_1)$ must be at least 1. In other words, $g_1^{(L,x)}(0; k_1) = 0$. So we may take $n \geq 1$. Now we have

$$\begin{aligned}
g_1^{(L,x)}(n; k_1) &= \mathbb{P}_x(N_1^{(L)}(k_1) = n) \\
&= \mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1} - 2) + \sum_{t=1}^{k_2-1} \mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1} - 1, \ldots, \\
&\qquad Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2) \\
&\quad + \mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1} - 1, Z_{T+2} = 2^{k_1} - 1, \ldots,
\end{aligned}$$

$$Z_{T+k_2-1} = 2^{k_1} - 1, Z_{T+k_2} = 2^{k_1} - 1). \tag{10}$$

We consider the terms in the summation first. For any $1 \leq t \leq k_2 - 1$, we have,

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1} - 1, Z_{T+2} = 2^{k_1} - 1, \ldots, \\
&\qquad Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2) \\
&= \mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1} - 1, Z_{T+2} = 2^{k_1} - 1, \ldots, \\
&\qquad Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2) \\
&\times \mathbb{P}_x(Z_{T+1} = 2^{k_1} - 1, Z_{T+2} = 2^{k_1} - 1, \ldots, \\
&\qquad Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2).
\end{aligned} \tag{11}$$

The second term in (11) can be written as

$$\begin{aligned}
&\mathbb{P}_x(Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2) \\
&= \mathbb{P}_x(Z_{T+t+1} = 2^{k_1} - 2 \mid Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+t} = 2^{k_1} - 1) \\
&\times \prod_{j=1}^{t} \mathbb{P}_x(Z_{T+j} = 2^{k_1} - 1 \mid Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+j-1} = 2^{k_1} - 1).
\end{aligned}$$

Now, for any $1 \leq j \leq t, T+j-1$ is also a stopping time. We denote by $\mathcal{F}_{T+j-1}$, the σ-algebra generated by the process Z_n up to the stopping time $T+j-1$, and by $\mathcal{F}_{(T+j-1)+}$, the σ-algebra generated by the process after the stopping time $T + j - 1$. Clearly, $\{Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+j-1} = 2^{k_1} - 1\} \in \mathcal{F}_{T+j-1}$ and $\{Z_{T+j} = 2^{k_1} - 1\} \in \mathcal{F}_{(T+j-1)+}$. Thus, by strong Markov property, we can write

$$\begin{aligned}
&\mathbb{P}_x(Z_{T+j} = 2^{k_1} - 1 \mid Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+j-1} = 2^{k_1} - 1) \\
&= \mathbb{P}_{Z_{T+j-1}}(Z_{T+j} = 2^{k_1} - 1) = \mathbb{P}_{2^{k_1}-1}(Z_1 = 2^{k_1} - 1) = p_{2^m-1}.
\end{aligned} \tag{12}$$

A similar argument shows that

$$\mathbb{P}_x(Z_{T+t+1} = 2^{k_1} - 2 \mid Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+t} = 2^{k_1} - 1) = q_{2^m-1}. \tag{13}$$

For the first term in (11), we note that $T+t+1$ is also a stopping time and $\{Z_{T+1} = 2^{k_1} - 1, \ldots, Z_{T+t} = 2^{k_1} - 1, Z_{T+t+1} = 2^{k_1} - 2\} \in \mathcal{F}_{T+t+1}$. Since $Z_{\tau_1(k_1)} = 2^{k_1} - 1$, we must have either $X_{T-k_1} = 0$ and $X_{T-j} = 1$ for $j = 0, 1, \ldots, k_1 - 1$ or $T = k_1$. Further, since $Z_{\tau_1(k_1)+j} = 2^{k_1} - 1$ for $j = 1, \ldots, t$ and $Z_{T+t+1} = 2^{k_1} - 2$, we also have $X_{T+j} = 1$ for $j = 0, 1, \ldots, t$ and $X_{T+t+1} = 0$. Therefore, we have a sequence of $\mathbf{1}'$s of length $k_1 + t$ with $t > 0$ which contributes to 1 run of length at least k_1 and since there are no runs of length k_1 before T, by the very definition of T, we have that the number of runs of length at least k_1 up to time $T + t + 1$ is 1. Since $t \leq k_2 - 1$, we have that $T + t + 1 < \tau_1(k, l)$. Let us define $Z'_i = Z_{i+T+t+1}$ for $i \geq 0$. Now, using the strong Markov property, we have that $\{Z'_i : i \geq 0\}$ is a homogeneous Markov chain with same transition matrix as that of $\{Z_i : i \geq 0\}$ with $Z'_0 = 2^{k_1} - 2$. Now, define $\tau'_1(k, l)$ as the stopping time for the process

$\{Z'_i : i \geq 0\}$. From the above discussion, we have that $\tau_1(k,l) = T + t + 1 + \tau'_1(k,l)$. Further, if we define, $N_1^{(L)'}(k_1)$ as the number runs of length at least k_1 up to time $\tau'_1(k,l)$ for the process $\{Z'_i : i \geq 0\}$, we must have that $N_1^{(L)'}(k_1) = n-1$. Therefore, we have,

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+t} = 2^{k_1}-1, Z_{T+t+1} = 2^{k_1}-2) \\
&= \mathbb{P}_{(2^m-2)}(N_1^{(L)'}(k_1) = n-1) = g_1^{(E,2^m-2)}(n-1;k_1).
\end{aligned} \tag{14}$$

Now, the first term in (10) can be written as

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1}-2) \\
&= \mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-2, Z_T = 2^{k_1}-1) \\
&\qquad \times \mathbb{P}_x(Z_{T+1} = 2^{k_1}-2 \mid Z_T = 2^{k_1}-1) \\
&= q_{2^m-1}\mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-2, Z_T = 2^{k_1}-1).
\end{aligned} \tag{15}$$

The arguments leading to equation (14) can now be repeated to conclude that

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-2, Z_T = 2^{k_1}-1) \\
&= \mathbb{P}_{(2^m-2)}(N_1^{(L)'}(k_1) = n-1) = g_1^{(E,2^m-2)}(n-1;k_1).
\end{aligned} \tag{16}$$

The last term in (10) can be similarly written as

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+k_2-1} = 2^{k_1}-1, Z_{T+k_2} = 2^{k_1}-1) \\
&= \prod_{j=1}^{k_2} \mathbb{P}_x(Z_{T+j} = 2^{k_1}-1 \mid Z_T = 2^{k_1}-1, Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+j-1} = 2^{k_1}-1) \\
&\qquad \times \mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_T = 2^{k_1}-1, Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+k_2} = 2^{k_1}-1) \\
&= \big(p_{2^m-1}\big)^{k_2}\mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+k_2} = 2^{k_1}-1).
\end{aligned}$$

Note that given $\{Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+k_2-1} = 2^{k_1}-1, Z_{T+k_2} = 2^{k_1}-1\}$, we have $\tau_1(k,l) = T + k_2$. Since this includes only run of length at least k_1, we have $N_1^{(L)}(k_1) = n$ if and only if $n = 1$. In other words, $\mathbb{P}_x(N_1^{(L)}(k_1) = n \mid Z_{T+1} = 2^{k_1}-1, \ldots, Z_{T+k_2-1} = 2^{k_1}-1, Z_{T+k_2} = 2^{k_1}-1) = \mathbb{I}_n(0)$ where $\mathbb{I}$ is the indicator function as defined in the statement of the Theorem.

Thus combining the above equation with equations (10) - (16), we can express

$$\begin{aligned}
g_1^{(L,x)}(n;k_1) &= q_{2^m-1}g_1^{(L,2^m-2)}(n-1;k_1) + \sum_{t=1}^{k_2-1} q_{2^m-1}\big(p_{2^m-1}\big)^t g_1^{(L,2^m-2)}(n-1;k_1) \\
&\qquad + \big(p_{2^m-1}\big)^{k_2}\mathbb{I}_n(1) \\
&= \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big) g_1^{(L,2^m-2)}(n-1;k_1) + \big(p_{2^m-1}\big)^{k_2}\mathbb{I}_n(1).
\end{aligned}$$

This completes the proof. ■

We note that the right hand side of (9) does not involve the initial condition $x \in N_m$. Therefore $g_1^{(L,x)}(n;k_1)$ must be independent of x. So, we will drop x and denote the above probability by $g_1^{(L)}(n;k_1)$.

Corollary 3 *For $n \geq 1$, we have*

$$g_1^{(L)}(n;k_1) = \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big)^{n-1} \big(p_{2^m-1}\big)^{k_2}. \tag{17}$$

Clearly, the equation (9) can be easily solved. Indeed, for $n = 1$, we have

$$g_1^{(L)}(1;k_1) = \big(p_{2^m-1}\big)^{k_2}$$

and for $n \geq 2$, we have

$$\begin{aligned} g_1^{(L)}(n;k_1) &= \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big) g_1^{(L)}(n-1;k_1) = \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big)^{n-1} g_1^{(L)}(1;k_1) \\ &= \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big)^{n-1} \big(p_{2^m-1}\big)^{k_2}. \end{aligned}$$

Hence the corollary follows.

We can deduce that $N_1^{(L)}(k_1) - 1$ follows a geometric distribution with parameter $\big(p_{2^m-1}\big)^{k_2}$ taking values $0, 1, \ldots$. Hence, the generating function of $N_1^{(L)}(k_1)$ is given by

$$\zeta_1^{(L)}(s;k_1) = \frac{\big(p_{2^m-1}\big)^{k_2} s}{1 - \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big) s}. \tag{18}$$

For $r > 1$, again, we can derive a similar result.

Theorem 3 *For $n \geq 1$ and $x \in A_m$, we have*

$$\begin{aligned} g_r^{(L,x)}(n;k_1) &= \big(p_{2^m-1}\big)^{k_2+(r-1)(l+1)} \mathbb{I}_n(1) + \Big(1 - \big(p_{2^m-1}\big)^{k_2}\Big) g_r^{(L,2^m-2)}(n-1;k_1) \\ &\quad + \big(p_{2^m-1}\big)^{k_2} \Big(1 - \big(p_{2^m-1}\big)^{l+1}\Big) \sum_{j=0}^{r-2} \big(p_{2^m-1}\big)^{j(l+1)} g_{r-1-j}^{(L,2^m-2)}(n-1;k_1) \end{aligned} \tag{19}$$

where $\mathbb{I}_{v_1}(v_2)$ is the indicator function, as defined in the previous theorem.

Proof : We follow the similar strategy as in the previous theorem. We consider the cases namely $k_2 = k - k_1 = 0$, i.e., $k = k_1$ and $k_2 > 0$, i.e., $k > k_1$ separately. We study the process from the time of the first occurrence of $k_1 (= k)$ consecutive successes, i.e., T occurs. Note there can be no occurrence of runs of length at least k_1 before this time, so that part of the process does not contribute to the counting of runs of length at least k_1. For $r \geq 2$, we have

$$g_r^{(L,x)}(n;k_1) = \mathbb{P}_x\big(N_1^{(L)}(k_1) = n, Z_{T+1} = 2^{k_1} - 1\big)$$

$$
\begin{aligned}
&+\sum_{t=1}^{(r-1)(l+1)-1} \mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1,\ldots,\\
&\qquad\qquad Z_{T+t}=2^{k_1}-1, Z_{T+t+1}=2^{k_1}-2\big)\\
&+\mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1, Z_{T+2}=2^{k_1}-1,\ldots,\\
&\qquad\qquad Z_{T+(r-1)(l+1)-1}=2^{k_1}-1, Z_{T+(r-1)(l+1)}=2^{k_1}-1\big).
\end{aligned}\tag{20}
$$

Similarly for $k_2>0$ and $r\geq 2$, from the time of first occurrence of k_1 consecutive successes, we have

$$
\begin{aligned}
g_r^{(L,x)}(n;k_1) &= \mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1\big)\\
&+\sum_{t=1}^{k_2-1} \mathbb{P}_x(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1,\ldots,\\
&\qquad\qquad Z_{T+t}=2^{k_1}-1, Z_{T+t+1}=2^{k_1}-2)\\
&+\sum_{t=k_2}^{k_2+(r-1)(l+1)-1} \mathbb{P}_x(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1,\ldots,\\
&\qquad\qquad Z_{T+t}=2^{k_1}-1, Z_{T+t+1}=2^{k_1}-2)\\
&+\mathbb{P}_x(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1, Z_{T+2}=2^{k_1}-1,\ldots,\\
&\qquad\qquad Z_{T+k_2+(r-1)(l+1)-1}=2^{k_1}-1, Z_{T+k_2+(r-1)(l+1)}=2^{k_1}-1).
\end{aligned}\tag{21}
$$

Now, the first term in both (20) and (21) are same and matches with the first term in (10) in the proof of Theorem 2. Therefore, same arguments, as in (15), provide the expression for this term. Indeed we have

$$
\begin{aligned}
&\mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1\big)\\
&=q_{2^m-1}\mathbb{P}_x\big(N_1^{(L)}(k_1)=n \mid Z_{T+1}=2^{k_1}-2\big)\\
&=q_{2^m-1}\mathbb{P}_{2^m-1}\big(N_1^{(L)}(k_1)=n-1\big)=q_{2^m-1}g_{r-1}^{(L,2^m-1)}(n-1;k_1).
\end{aligned}\tag{22}
$$

The terms in summation in both (20) and (21) are similar to the terms in summation in (10) in the proof of Theorem 2. However, there is a small difference in the first summation in (21). In this case, no runs of length k will be completed as the index runs from 1 to k_2-1 and we need at least $k_2=k-k_1$ successes after the occurrence of the first consecutive k_1 successes to complete a run of length k. Therefore, for the terms in first summation in (21), we have

$$
\begin{aligned}
&\mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1,\ldots, Z_{T+t}=2^{k_1}-1, Z_{T+t+1}=2^{k_1}-2\big)\\
&=q_{2^m-1}\big(p_{2^m-1}\big)^t g_r^{(L,2^m-2)}(n-1;k_1).
\end{aligned}\tag{23}
$$

For the other term in summation in (21) and the terms in the summation (20), at least one l-look-back run will be completed. It is easy to check that the exact number of

l-look-back runs completed is given by $\lfloor (t-k_2)/(l+1) \rfloor$ where $\lfloor a \rfloor$ is the largest integer smaller or equal to a. Thus, we have

$$\begin{aligned}
&\mathbb{P}_x(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1,\dots,Z_{T+t}=2^{k_1}-1, Z_{T+t+1}=2^{k_1}-2) \\
&= q_{2^m-1}\big(p_{2^m-1}\big)^t g_i^{(L,2^m-2)}(n-1;k_1) \qquad (24)
\end{aligned}$$

where $i=\lfloor (t-k_2)/(l+1) \rfloor$.

The final terms in both cases are similar and can be dealt in the exact same way as done for the last term in previous Theorem. Indeed, the same calculations yield

$$\begin{aligned}
&\mathbb{P}_x\big(N_1^{(L)}(k_1)=n, Z_{T+1}=2^{k_1}-1, Z_{T+2}=2^{k_1}-1,\dots, \\
&\qquad Z_{T+k_2+(r-1)(l+1)-1}=2^{k_1}-1, Z_{T+k_2+(r-1)(l+1)}=2^{k_1}-1\big) \\
&= \big(p_{2^m-1}\big)^{k_2+(r-1)(l+1)}\mathbb{I}_n(1). \qquad (25)
\end{aligned}$$

Now, for $k_2=0$, using (20) and combining the equations (22), (23), (24) and (25), we obtain

$$\begin{aligned}
&g_r^{(L,x)}(n;k_1) \\
&= \sum_{j_1=0}^{r-2}\sum_{j_2=0}^{l} q_{2^m-1}\big(p_{2^m-1}\big)^{j_2+j_1(l+1)} g_{r-1-j_1}^{(L,2^m-2)}(n-1;k_1) + \big(p_{2^m-1}\big)^{(r-1)(l+1)}\mathbb{I}_n(1) \\
&= \Big(1-\big(p_{2^m-1}\big)^{l+1}\Big)\sum_{j=0}^{r-2}\big(p_{2^m-1}\big)^{j(l+1)} g_{r-1-j}^{(L,2^m-2)}(n-1;k_1) + \big(p_{2^m-1}\big)^{(r-1)(l+1)}\mathbb{I}_n(1).
\end{aligned}$$

For $k_2>0$, using (21) and combining the equations (22), (24) and (25), we obtain

$$\begin{aligned}
g_r^{(L,x)}(n;k_1) &= \sum_{t=0}^{k_2-1} q_{2^m-1}\big(p_{2^m-1}\big)^t g_r^{(L,2^m-2)}(n-1;k_1) \\
&+ \sum_{j_1=0}^{r-2}\sum_{j_2=0}^{l} q_{2^m-1}\big(p_{2^m-1}\big)^{k_2+j_1(l+1)+j_2} g_{r-1-j_1}^{(L,2^m-2)}(n-1;k_1) + \big(p_{2^m-1}\big)^{k_2+(r-1)(l+1)}\mathbb{I}_n(1) \\
&= \Big(1-\big(p_{2^m-1}\big)^{k_2}\Big) g_r^{(L,2^m-2)}(n-1;k_1) \\
&\quad + \big(p_{2^m-1}\big)^{k_2}\Big(1-\big(p_{2^m-1}\big)^{l+1}\Big)\sum_{j=0}^{r-2}\big(p_{2^m-1}\big)^{j(l+1)} g_{r-1-j}^{(L,2^m-2)}(n-1;k_1) \\
&\quad + \big(p_{2^m-1}\big)^{k_2+(r-1)(l+1)}\mathbb{I}_n(1).
\end{aligned}$$

This completes the proof. ∎

We first show that $g_r^{(L,x)}(\cdot;k_1)$ independent of $x \in A_m$. We have proved this for $r=1$ (see Corollary 3). Now, assume that $g_r^{(L,x)}(\cdot;k_1)$ independent of $x \in A_m$. Using equations (19), we have that $g_{r+1}^{(L,x)}(\cdot;k_1)$ can be expressed as weighted sums of $g_i^{(L,x)}(\cdot;k_1)$

for $i = 1, 2, \ldots, r$. Since the right hand side of the equation (19) does not involve any $x \in A_m$, $g_{r+1}^{(L,x)}(\cdot; k_1)$ must be independent of x. Therefore, from now on we will drop the superscript x from the notation and denote it by $g_r^{(L)}(\cdot; k_1)$. This may be summarised as follows:

Lemma 1 *For any $x \in A_m$ and $r \geq 1$, the probability $g_r^{(L,x)}(n; k_1) = \mathbb{P}_x(N_r^{(L)}(k_1) = n)$ is independent of x. Further, for $r \geq 2$, $g_r^{(L,x)}(n; k_1)$ satisfies the recurrence relation*

$$\begin{aligned} &g_r^{(L)}(n; k_1) \\ &= \left(p_{2^m-1}\right)^{k_2+(r-1)(l+1)}\mathbb{I}_n(1) + \left(1 - \left(p_{2^m-1}\right)^{k_2}\right) g_r^{(L)}(n-1; k_1) \\ &+ \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right) \sum_{j=0}^{r-2} \left(p_{2^m-1}\right)^{j(l+1)} g_{r-1-j}^{(L)}(n-1; k_1). \end{aligned} \tag{26}$$

Next we concentrate on deriving the generating function of $\{g_r^{(L)}(n; k_1) : n \geq 0\}$. Now, using the relation (26), we can easily develop the recurrence relation between the generating functions of $N_r^{(L)}(k_1)$. Let us denote the probability generating function of $N_r^{(L)}(k_1)$ by $\zeta_r^{(L)}(s; k_1)$, i.e.,

$$\zeta_r^{(L)}(s; k_1) = \sum_{n=0}^{\infty} s^k g_r^{(L)}(n; k_1).$$

The probability generating function $\zeta_r^{(L)}(s; k_1)$, for $r \geq 2$ and $k_2 \geq 0$, is given by

$$\begin{aligned} \zeta_r^{(L)}(s; k_1) &= \left(p_{2^m-1}\right)^{k_2+(r-1)(l+1)} s + \sum_{n=1}^{\infty}\left(1 - \left(p_{2^m-1}\right)^{k_2}\right) g_r^{(L)}(n-1; k_1) s^n \\ &\quad + \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right) \sum_{n=0}^{\infty}\sum_{j=0}^{r-2} \left(p_{2^m-1}\right)^{j(l+1)} g_{r-1-j}^{(L)}(n-1; k_1) s^n \\ &= \left(p_{2^m-1}\right)^{k_2+(r-1)(l+1)} s + \left(1 - \left(p_{2^m-1}\right)^{k_2}\right) s \zeta_r^{(L)}(s; k_1) \\ &\quad + \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right) s \sum_{j_1=0}^{r-2} \left(p_{2^m-1}\right)^{j_1(l+1)} \zeta_{r-1-j_1}^{(L)}(s; k_1). \end{aligned}$$

Simplifying, we obtain the following lemma.

Lemma 2 *For $r \geq 2$, the sequence of probability generating functions $\{\zeta_r^{(L)}(s; k_1) : r \geq 1\}$ satisfies the recurrence relation*

$$\begin{aligned} &\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right) s\right] \zeta_r^{(L)}(s; k_1) \\ &= \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right) s \sum_{j_1=0}^{r-2} \left(p_{2^m-1}\right)^{j_1(l+1)} \zeta_{r-1-j_1}^{(L)}(s; k_1) \\ &\quad + \left(p_{2^m-1}\right)^{k_2+(r-1)(l+1)} s. \end{aligned} \tag{27}$$

Proof of Theorem 1: Let us define by $\Xi^{(L)}(z;k_1)$ the generating function of the sequence $\{\zeta_r^{(L)}(s;k_1) : r \geq 1\}$, i.e.,

$$\Xi^{(L)}(z;k_1) = \sum_{r=1}^{\infty} \zeta_r^{(L)}(s;k_1) z^r.$$

Now, using Lemma 2 and expression in (18), consider

$$\begin{aligned}
&\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]\Xi^{(L)}(z;k_1) \\
&= \left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]\zeta_1^{(L)}(s;k_1)z + \sum_{r=2}^{\infty} s\left(p_{2^m-1}\right)^{k_2+(r-1)(l+1)} z^r \\
&\quad + \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)s \sum_{r=2}^{\infty}\sum_{j=0}^{r-2}\left(p_{2^m-1}\right)^{j(l+1)}\zeta_{r-1-j}^{(L)}(s;k_1)z^r \\
&= \left(p_{2^m-1}\right)^{k_2} sz + \left(p_{2^m-1}\right)^{k_2} sz \sum_{r=1}^{\infty}\left(p_{2^m-1}\right)^{r(l+1)} z^r \\
&\quad + \left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)s \sum_{j=0}^{\infty}\sum_{r=j}^{\infty}\left(p_{2^m-1}\right)^{j(l+1)}\zeta_{r-j+1}^{(L)}(s;k_1)z^{r+2} \\
&= \frac{\left(p_{2^m-1}\right)^{k_2} sz}{1 - \left(p_{2^m-1}\right)^{(l+1)} z} + sz\Xi^{(L)}(z;k_1)\left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)\sum_{j=0}^{\infty}\left(p_{2^m-1}\right)^{j(l+1)} z^j \\
&= \frac{\left(p_{2^m-1}\right)^{k_2} sz}{1 - \left(p_{2^m-1}\right)^{(l+1)} z} + \frac{sz\Xi^{(L)}(z;k_1)\left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)}{1 - \left(p_{2^m-1}\right)^{(l+1)} z}.
\end{aligned} \tag{28}$$

Now the above equation (28) yields the expression of $\Xi^{(L)}(z;k_1)$ as

$$\begin{aligned}
\Xi^{(L)}(z;k_1) &= \left[\left(p_{2^m-1}\right)^{k_2} sz\right]\Bigg[\left(1 - \left(p_{2^m-1}\right)^{l+1} z\right)\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right] \\
&\qquad - sz\left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)\Bigg]^{-1} \\
&= \left[\left(p_{2^m-1}\right)^{k_2} sz\right]\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]^{-1} \\
&\qquad \times \left[1 - \left(p_{2^m-1}\right)^{l+1} z - \frac{sz\left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)}{\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]}\right]^{-1} \\
&= z\left[\left(p_{2^m-1}\right)^{k_2} s\right]\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]^{-1} \\
&\qquad \times \left[1 - z\left(\left(p_{2^m-1}\right)^{l+1} + \frac{s\left(p_{2^m-1}\right)^{k_2}\left(1 - \left(p_{2^m-1}\right)^{l+1}\right)}{\left[1 - \left(1 - \left(p_{2^m-1}\right)^{k_2}\right)s\right]}\right)\right]^{-1}
\end{aligned} \tag{29}$$

The coefficient of z^r yields the expression for $\zeta_r^{(L)}(s;k_1)$. Thus, we see that coefficient of z^r is given by multiplying the coefficient of z^{r-1} in the last line of equation (29) by $\left[\left(p_{2^m-1}\right)^{k_2} s\right]\left[1-\left(1-\left(p_{2^m-1}\right)^{k_2}\right)s\right]^{-1}$. Using the expansion $(1-az)^{-1} = \sum_{n=0}^{\infty}(az)^n = \sum_{n=0}^{\infty} a^n z^n$, we have

$$
\begin{aligned}
&\zeta_r^{(L)}(s;k_1)\\
&= \left[\left(p_{2^m-1}\right)^{k_2} s\right]\left[1-\left(1-\left(p_{2^m-1}\right)^{k_2}\right)s\right]^{-1}\\
&\qquad \times \left[\left(p_{2^m-1}\right)^{l+1} + \frac{s\left(p_{2^m-1}\right)^{k_2}\left(1-\left(p_{2^m-1}\right)^{l+1}\right)}{\left[1-\left(1-\left(p_{2^m-1}\right)^{k_2}\right)s\right]}\right]^{r-1}\\
&= \left[\frac{\left(p_{2^m-1}\right)^{k_2} s}{1-\left(1-\left(p_{2^m-1}\right)^{k_2}\right)s}\right]\left[\left(p_{2^m-1}\right)^{l+1} + \frac{\left(p_{2^m-1}\right)^{k_2} s}{1-\left(1-\left(p_{2^m-1}\right)^{k_2}\right)s}\left(1-\left(p_{2^m-1}\right)^{l+1}\right)\right]^{r-1}.
\end{aligned}
$$

This completes the proof of the theorem. ∎

Acknowledgement: The author wishes to thank the referees for their helpful comments which have improved the presentation of the paper.

References

Aki S. and Hirano K. (1994) *Distributions of numbers of failures and successes until the first consecutive k successes*, *Ann. Inst. Statist. Math.*, **46**, 193-202.

Aki S. and Hirano K. (1995) *Joint distributions of numbers of success-runs and failures until the first k consecutive successes*, *Ann. Inst. Statist. Math.*, **47**, 225-235.

Anuradha (2022a) *Asymptotic results for generalized runs in higher order Markov Chains*, *Statistics and Applications*, (to appear).

Anuradha (2022b) *Distribution of runs of length exactly k_1 until a stopping time for higher order Markov chain.* (preprint).

Ebneshahrashoob M. and Sobel M. (1990) *Sooner or Later waiting time problems for Bernoulli trials: Frequency and run quotas*, *Statist. Probab. Lett.*, **9**, 5-11.

Feller W. (1968) *An introduction to Probability theory & its applications. Vol - I.* (John Wiley, New York, 3rd ed.)

Fu J. C. and Koutras M. V. (1994) *Distribution theory of runs: a Markov chain approach*, *J. Amer. Statist. Assoc.*, **89**, 1050-1058.

Hirano K., Aki S. and Uchida M. (1997) *Distributions of numbers of success-runs until the first consecutive k successes in higher order Markov dependent trials*, *Advances* *in combinatorial methods and applications to probability and statistics,* 401–410, Stat. Ind. Technol., (Birkhäuser Boston, Boston, MA).

Martin D. E. K. (2005) *Distribution of the number of successes in success runs of length at least k for a higher order Markovian sequence*, *Methodology and Computing in Applied Probability,* **7** 543–554.

Uchida M. (1998) *On number of occurrences of success runs of specified length in a higher order two-state Markov chain*, *Ann. Inst. Statist. Math.*, **50**, 587-601.